Solutions to Challenge Corner of 3rd issue of IMOment

1. Let $a_1, a_2, a_3, \ldots$ be a sequence of numbers. Given $a_1 = 2$ and for all positive integers *n*, $a_{n+1} = a_n + 2^n + 1$, find a_{2015} in terms of *n*.

   Solution: The answer is $a_n = 2^n + n - 1$, which can be proved by mathematical induction.

2. A positive integer is said to be a **cow number** if its digits consist of alternate 1's and 8's. For example, 1, 8, 18, 181 and 8181 are cow numbers, whereas 79, 11 and 1881 are not cow numbers. Find all cow numbers that are perfect squares.

   Solution: Note that no perfect square has units digit 8. Among the cow numbers with units digit 1, note that 1 and 81 are perfect squares. The other cow numbers with units digit 1 end in 181 and therefore leave a remainder of 5 when divided by 8 (since 1000 is divisible by 8). However, the square of any odd integer leaves a remainder of 1 when divided by 8. This is because for any integer *n*,

$$(2n+1)^2 = 4n^2 + 4n + 1 = 4n(n+1) + 1,$$

   where one of *n* and $n+1$ is even and hence is $4n(n+1)$ divisible by 8. In conclusion, the only cow numbers that are perfect squares are 1 and 81.

3. Suppose $ABCD$ is an isosceles trapezium with $AB//DC$ and $AD = BC$, and there is a circle in the trapezium that is tangent to all four sides (i.e. that touches each side at exactly one point). Suppose the circle touches BC at E and touches AD at F. Prove that the lines AC, BD and EF meet at one point.

Solution: Let G be the point of intersection of the lines AC and EF. In order to show that G is on BD, it is sufficient to show that G is the midpoint of EF, as a similar argument would yield that BD bisects EF. We compute EG first. Since triangles ABC and GEC are similar, we have $EG = (AB \cdot CE)/BC$. But $CE = CD/2$, so $EG = (AB \cdot CD)/(2BC)$. For FG, we consider the similar triangles ACD and AGF, and we have $FG = (CD \cdot AF)/AD$. But $AF = AB/2$, so $FG = (CD \cdot AB)/(2AD) = (AB \cdot CD)/(2BC) = EG$, as desired.

4. Here is a "proof" that all people have the same height: "Let $P(n)$ be the statement that whenever n people are chosen, they always have the same height. Clearly, $P(1)$ is true. Suppose $P(k)$ is true. Then whenever $(k + 1)$ people are chosen, the inductive hypothesis tells us that the 1st, 2nd, …, kth people have the same height, and also that the 2nd, 3rd, …, $(k + 1)$th people have the same height. Hence, all the k+1 people have the same height. So $P(k + 1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers n." What is wrong with this proof? (You may refer to the article on mathematical induction in this issue.)

Solution: The derivation of $P(k + 1)$ from $P(k)$ given in the problem statement is not applicable to the case of $k = 1$.

List of awardees:

Name of student	Name of school
CHEUNG Kai-hei Trevor	St. Paul’s Co-educational College
FONG Tsz-lo	SKH Lam Woo Memorial Secondary School
Laurance LAU	Diocesan Boys’ School

Awardees will also be notified with a separate e-mail.